

NOETHER CURRENTS FOR HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ TYPE WITH TIME DELAY

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ABSTRACT. We study, from an optimal control perspective, Noether currents for higher-order problems of Herglotz type with time delay. Main result provides new Noether currents for such generalized variational problems, which are particularly useful in the search of extremals. The proof is based on the idea of rewriting the higher-order delayed generalized variational problem as a first-order optimal control problem without time delays.

1. Introduction. This article is devoted to the proof of a second Noether type theorem for higher-order delayed variational problems of Herglotz. Such problems, which are invariant under a certain group of transformations, were first studied in 1918 by Emmy Noether for the particular case of first-order variational problems without time delay [22]. In her famous paper [22], Noether proved two remarkable theorems that relate the invariance of a variational integral with properties of its Euler–Lagrange equations. Since most physical systems can be described by using Lagrangians and their associated actions, the importance of Noether’s two theorems is obvious [3].

The first Noether’s theorem, usually simply called Noether’s theorem, ensures the existence of r conserved quantities along the Euler–Lagrange extremals when the variational integral is invariant with respect to a continuous symmetry transformation that depend on r parameters [34]. Noether’s theorem explains all conservation laws of mechanics, for instance, invariance under translation in time implies conservation of energy; conservation of linear momentum comes from invariance of the system under spacial translations; invariance under rotations in the base space yields conservation of angular momentum.

The second Noether’s theorem, less known than the first one, applies to variational problems that are invariant under a certain group of transformations that depends on arbitrary functions and their derivatives up to some order [32]. In contrast to Noether’s theorem, where the transformations are global, in second Noether’s theorem the transformations are local: they can affect every part of the system differently. Noether’s second theorem has applications in several fields, such as, general relativity, hydromechanics, electrodynamics, and quantum chromodynamics [8, 18, 28]. Extensions of both Noether’s theorems to optimal control problems were first obtained in [30, 31, 32, 33]. For systems with time delay, see [6]. In 2013, the second Noether theorem was extended to the context of fractional calculus [19] and time scales [20].

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Motivated by the important applications of Noether's second theorem [20] and the applicability of higher-order dynamic systems with time delays in modeling real-life phenomena [4, 7, 29], as well as the importance of variational problems of Herglotz [14, 16], our goal in this paper is to study generalized variational problems that are invariant under a certain group of transformations that depends on arbitrary functions and their derivatives up to some order, and deduce expressions for Noether currents, that is, expressions that are constant in time along the extremals.

Our work is related with the second Noether theorem for optimal control in the sense of [32], and is particularly useful because provides necessary conditions for the search of extremals. There are other different results on the calculus of variations, also related with the notion of invariance under a certain group of transformations that depends on arbitrary functions and their derivatives, but they are concerned with Noether identities [10, 20, 21] and not with Noether currents as we do here.

The generalized variational problem was introduced by Herglotz in 1930 [16], and consists in the determination of $x \in C^1([a, b]; \mathbb{R}^m)$ and $z \in C^1([a, b]; \mathbb{R})$, such that

$$\begin{aligned} z(b) &\longrightarrow \text{extr}, \\ \dot{z}(t) &= L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a, b], \\ \text{subject to } x(a) &= \alpha \quad \text{and} \quad z(a) = \gamma \end{aligned} \tag{H^1}$$

for some $\alpha \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}$, where by “extr” we mean “to minimize or maximize” and the Lagrangian $L \in C^1([a, b] \times \mathbb{R}^{2m+1}; \mathbb{R})$ is such that $t \mapsto \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t))$, $t \mapsto \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t))$ and $t \mapsto \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))$ are differentiable. It is clear that if the Lagrangian L does not depend on the variable z , then we get the classical problem of the calculus of variations. The variational problem of Herglotz attracted the interest of the mathematical community in the last two decades, after the publications [13, 14]. Namely, the two Noether theorems were proved for the first-order problem in [9, 10, 11]. The first Noether theorem for variational problems of Herglotz type with time delay was proved in [25]. The higher-order problem of Herglotz was introduced in [24]. Noether's first theorem for higher-order problems was proved in [26] and, more recently, using an optimal control approach, the authors generalized previous results for higher-order problems with time delay in [27]. The variational problem of Herglotz was also considered in the context of fractional calculus in [2] and, in the general context of Riemannian manifolds, in [1].

The manuscript is organized as follows. In Section 2, we present the results that constitute the basis of our work: a version of Pontryagin's maximum principle, higher-order delayed Euler–Lagrange equations and Noether's second theorem for optimal control problems. In Section 3, we prove our main results: a second Noether theorem for higher-order problems of Herglotz with time delay (Theorem 3.1) and two important corollaries: the first (Corollary 1) is devoted to first-order variational problems of Herglotz with time delay, while the second (Corollary 2) is devoted to first-order classical variational problems with time delay. We finish the paper with an illustrative example (Section 4) and concluding remarks (Section 5).

2. Preliminaries. In this paper we consider the following generalized variational problem (\mathbf{H}_τ^n) .

Problem (\mathbf{H}_τ^n) . Let τ be a real number such that $0 \leq \tau < b - a$. Determine piecewise trajectories $x \in PC^n([a - \tau, b]; \mathbb{R}^m)$ and a function $z \in PC^1([a, b]; \mathbb{R})$

such that:

$$z(b) \longrightarrow \text{extr},$$

where the pair $(x(\cdot), z(\cdot))$ satisfies the differential equation

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), \dots, x^{(n)}(t), x(t-\tau), \dot{x}(t-\tau), \dots, x^{(n)}(t-\tau), z(t)\right),$$

for $t \in [a, b]$, and is subject to initial conditions

$$z(a) = \gamma \in \mathbb{R} \quad \text{and} \quad x^{(k)}(t) = \mu^{(k)}(t), \quad k = 0, \dots, n-1,$$

where $\mu \in PC^n([a-\tau, a]; \mathbb{R}^m)$ is a given initial function. The Lagrangian L is assumed to satisfy the following hypotheses:

- i. $L \in C^1([a, b] \times \mathbb{R}^{2m(n+1)}; \mathbb{R})$;
- ii. functions $t \mapsto \frac{\partial L}{\partial z}[x; z]_\tau^n(t)$, $t \mapsto \frac{\partial L}{\partial x^{(k)}}[x; z]_\tau^n(t)$ and $t \mapsto \frac{\partial L}{\partial x_\tau^{(k)}}[x; z]_\tau^n(t)$ are differentiable for any admissible pair $(x(\cdot), z(\cdot))$, $k = 0, \dots, n$,

where, to simplify expressions, we use the notation $x_\tau^{(k)}(t)$ to denote the k th derivative of x evaluated at $t-\tau$ (often we use $x_\tau(t)$ for $x_\tau^{(0)}(t) = x(t-\tau)$ and $\dot{x}_\tau(t)$ for $x_\tau^{(1)}(t) = \dot{x}(t-\tau)$) and

$$[x; z]_\tau^n(t) := \left(t, x(t), \dot{x}(t), \dots, x^{(n)}(t), x_\tau(t), \dot{x}_\tau(t), \dots, x_\tau^{(n)}(t), z(t)\right).$$

Associated with the generalized variational problem (\mathbf{H}_τ^n) , one has the following definitions.

Definition 2.1 (Admissible pair to problem (\mathbf{H}_τ^n)). We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in PC^n([a-\tau, b]; \mathbb{R}^m)$ and $z(\cdot) \in PC^1([a, b]; \mathbb{R})$ is an admissible pair to problem (\mathbf{H}_τ^n) if it satisfies the equation

$$\dot{z}(t) = L[x; z]_\tau^n(t), \quad t \in [a, b],$$

subject to

$$z(a) = \gamma \quad \text{and} \quad x^{(k)}(t) = \mu^{(k)}(t)$$

for all $k = 0, 1, \dots, n-1$, $t \in [a-\tau, a]$ and $\gamma \in \mathbb{R}$.

Definition 2.2 (Extremizer to problem (\mathbf{H}_τ^n)). An admissible pair $(x^*(\cdot), z^*(\cdot))$ is said to be an extremizer to problem (\mathbf{H}_τ^n) if $z(b) - z^*(b)$ has the same signal for all admissible pairs $(x(\cdot), z(\cdot))$ that satisfy $\|z - z^*\|_0 < \epsilon$ and $\|x - x^*\|_0 < \epsilon$ for some positive real ϵ , where $\|y\|_0 = \max_{a \leq t \leq b} |y(t)|$.

Inspired by the ideas presented in [15] (see also [5, 12, 17, 27]), problem (\mathbf{H}_τ^n) can be rewritten as a first-order optimal control problem without time delay. Such reduction is presented in Section 3. Firstly, let us recall some key notions and results from optimal control theory. Consider the optimal control problem in Bolza form on the interval $[a, b]$:

$$\begin{aligned} \mathcal{J}(x(\cdot), u(\cdot)) &= \int_a^b f(t, x(t), u(t)) dt + \phi(x(b)) \longrightarrow \text{extr} \\ \text{subject to } \dot{x}(t) &= \varphi(t, x(t), u(t)), \end{aligned} \tag{P}$$

with some initial condition on x , where $f \in C^1([a, b] \times \mathbb{R}^m \times \Omega; \mathbb{R})$, $\phi \in C^1(\mathbb{R}^m; \mathbb{R})$, $\varphi \in C^1([a, b] \times \mathbb{R}^m \times \Omega; \mathbb{R}^m)$, $x \in PC^1([a, b]; \mathbb{R}^m)$ and $u \in PC([a, b]; \Omega)$, with $\Omega \subseteq \mathbb{R}^r$ an open set. Function x is called the state variable and u the control variable; ϕ is known as the payoff term.

A fundamental tool in optimal control theory is the well-known Pontryagin's maximum principle.

Theorem 2.3 (Pontryagin's maximum principle for problem (P) [23]). *If a pair $(x(\cdot), u(\cdot))$ with $x \in PC^1([a, b]; \mathbb{R}^m)$ and $u \in PC([a, b]; \Omega)$ is a solution to problem (P) with the initial condition $x(a) = \alpha$, $\alpha \in \mathbb{R}^m$, then there exists a multiplier $\psi \in PC^1([a, b]; \mathbb{R}^m)$ such that for the Hamiltonian H defined by*

$$H(t, x, u, \psi) := f(t, x, u) + \psi \cdot \varphi(t, x, u) \quad (1)$$

the next conditions hold:

- the optimality condition

$$\frac{\partial H}{\partial u}(t, x(t), u(t), \psi(t)) = 0; \quad (2)$$

- the adjoint system

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi}(t, x(t), u(t), \psi(t)) \\ \dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi(t)); \end{cases} \quad (3)$$

- the transversality condition

$$\psi(b) = \text{grad}(\phi(x))(b). \quad (4)$$

The following definition is of central importance for the formulation of second Noether's theorem.

Definition 2.4 (Noether current [32]). A function $C(t, x(t), u(t), \psi(t))$, which is constant along every $x \in PC^1([a, b]; \mathbb{R}^m)$, $u \in PC([a, b]; \Omega)$ and $\psi \in PC^1([a, b]; \mathbb{R}^m)$ solution of (2)–(4), is called a Noether current.

In order to apply the results of [32] to the Bolza problem (P), we rewrite it in the following equivalent Lagrange form:

$$\begin{aligned} \mathcal{I}(x(\cdot), y(\cdot), u(\cdot)) &= \int_a^b (f(t, x(t), u(t)) + y(t)) dt \longrightarrow \text{extr} \\ \text{subject to } &\begin{cases} \dot{x}(t) = \varphi(t, x(t), u(t)), \\ \dot{y}(t) = 0, \end{cases} \end{aligned}$$

$$\text{and to the initial conditions } x(a) = \alpha \text{ and } y(a) = \frac{\phi(x(b))}{b-a}.$$

Using Pontryagin's maximum principle, the following result follows [27].

Theorem 2.5 (Higher-order delayed Euler–Lagrange equations and transversality conditions [27]). *If $(x(\cdot), z(\cdot))$ is an extremizer to problem (\mathbf{H}_τ^n) that satisfies the conditions $x^{(k)}(t) = \mu^{(k)}(t)$, with $\mu \in PC^n([a - \tau, a]; \mathbb{R}^m)$, $k = 0, \dots, n - 1$ and $t \in [a - \tau, a]$, then the following two Euler–Lagrange equations hold:*

$$\sum_{l=0}^n (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x; z]_\tau^n(t) + \psi_z(t + \tau) \frac{\partial L}{\partial x_\tau^{(l)}}[x; z]_\tau^n(t + \tau) \right) = 0,$$

for $t \in [a, b - \tau]$, and

$$\sum_{l=0}^n (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l)}}[x; z]_\tau^n(t) \right) = 0,$$

for $t \in [b - \tau, b]$, where ψ_z is defined by

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x; z]_\tau^n(\theta) d\theta}, \quad t \in [a, b].$$

Furthermore, the following transversality conditions are satisfied:

$$\sum_{l=0}^{n-k} (-1)^l \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x; z]_\tau^n(t) \right) \Big|_{t=b} = 0, \quad k = 1, \dots, n.$$

In addition to previous result, we were able to obtain in [27] expressions for the multipliers related to z and x , and also the expression of the Hamiltonian of problem (\mathbf{H}_τ^n) . They are, respectively:

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}[x; z]_\tau^n(\theta) d\theta}, \quad t \in [a, b], \quad (5)$$

$$\begin{aligned} \phi_k(t) &= \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l} \left(\psi_z(t+\tau) \frac{\partial L}{\partial x_\tau^{(l+k)}}[x; z]_\tau^n(t+\tau) \right), \quad t \in [a-\tau, a], \\ \phi_k(t) &= \sum_{l=0}^{n-k} (-1)^{l+1} \frac{d^l}{dt^l} \left(\psi_z(t) \frac{\partial L}{\partial x^{(l+k)}}[x; z]_\tau^n(t) + \psi_z(t+\tau) \frac{\partial L}{\partial x_\tau^{(l+k)}}[x; z]_\tau^n(t+\tau) \right), \end{aligned} \quad (6)$$

$t \in [a, b]$, and

$$H = \sum_{k=1}^n \phi_k(t) \cdot x^{(k)}(t) + \psi_z(t) L[x; z]_\tau^n(t), \quad t \in [a, b]. \quad (7)$$

Before presenting Noether's second theorem for the optimal control (P), we need to introduced a notion of invariance. In this paper we follow the definition of semi-invariance presented in [32].

Definition 2.6 (Semi-invariance of problem (P) under a group of symmetries [32]). Let $p : [a, b] \rightarrow \mathbb{R}^d$ be an arbitrary function of class C^q . Using the notation

$$\alpha(t) := \left(t, x(t), u(t), p(t), \dot{p}(t), \dots, p^{(q)}(t) \right),$$

we say that the optimal control problem (P) is semi-invariant if there exist a C^1 transformation group

$$\begin{aligned} g : [a, b] \times \mathbb{R}^m \times \Omega \times \mathbb{R}^{d \times (q+1)} &\rightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r, \\ g(\alpha(t)) &= (\mathbf{T}(\alpha(t)), \mathbf{X}(\alpha(t)), \mathbf{U}(\alpha(t))), \end{aligned} \quad (8)$$

which for $p(t) = \dot{p}(t) = \dots = p^{(q)}(t) = 0$ coincides with the identity transformation for all $(t, x, u) \in [a, b] \times \mathbb{R}^m \times \Omega$, satisfying the following conditions:

$$\begin{aligned} &\left(\theta_0 \cdot p(t) + \theta_1 \cdot \dot{p}(t) + \dots + \theta_q \cdot p^{(q)}(t) \right) \frac{d}{dt} f(t, x(t), u(t)) + f(t, x(t), u(t)) \\ &+ \frac{\phi(x(b))}{b-a} + \frac{d}{dt} F(\alpha(t)) = \left(f(g(\alpha(t))) + \frac{\phi(X(\alpha(b)))}{T(\alpha(b)) - T(\alpha(a))} \right) \frac{d}{dt} \mathbf{T}(\alpha(t)), \\ &\frac{d}{dt} \mathbf{X}(\alpha(t)) = \varphi(g(\alpha(t))) \frac{d}{dt} \mathbf{T}(\alpha(t)), \end{aligned}$$

for some function F of class C^1 and some $\theta_0, \dots, \theta_q \in \mathbb{R}^d$.

Remark 1. The group of transformations g (8) is usually called a gauge symmetry of the optimal control problem, in order to emphasize the fact that the transformations depend on arbitrary functions and, therefore, have local nature.

Theorem 2.7 (Noether's second theorem for the optimal control problem (P) [32]). *If problem (P) is semi-invariant under a group of symmetries as in Definition 2.6, then there are $d(q+1)$ Noether currents of the form*

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \left(f(t, x(t), u(t) + \frac{\phi(x(b))}{b-a}) \right. \\ \left. + \psi(t) \cdot \frac{\partial X(\alpha(t))}{\partial p_J^{(I)}} \right|_0 - H(t, x(t), u(t), \psi(t)) \frac{\partial T(\alpha(t))}{\partial p_J^{(I)}} \right|_0 \end{aligned}$$

for $I = 0, \dots, q$, $J = 1, \dots, d$, where H is defined in (1) and $(*)|_0$ stands for $(*)|_{p(t)=\dot{p}(t)=\dots=p^{(q)}(t)=0}$.

Remark 2. It is clear that if $\varphi = u$, $\theta_0 = \dots = \theta_q = 0$ and $F \equiv 0$, and the transformation group g does not depend on the derivatives of the state variables, then Theorem 2.7 reduces to the classical Noether's second theorem for the basic problem of the calculus of variations.

3. Proof of main result. The central idea of the proof of our main result, Noether's second theorem for the higher-order variational problem of Herglotz type with time delay, is to rewrite problem (\mathbf{H}_τ^n) as a non-delayed optimal control problem. For this, we assume, without loss of generality, that the initial time is zero ($a = 0$) and the final time b is an integer multiple of τ , that is, $b = N\tau$ for some $N \in \mathbb{N}$ (see Remark 3). Therefore, we can divide the interval $[a, b]$ into N equal parts. Fix $t \in [0, \tau]$ and introduce variables $x^{k;i}$ and z_j with $k = 0, \dots, n$, $i = 0, \dots, N$, and $j = 1, \dots, N+1$, as follows:

$$\begin{aligned} x^{k;i}(t) &= x^{(k)}(t + (i-1)\tau), \quad z_j(t) = z(t + (j-1)\tau), \\ \dot{z}_j(t) &= L_j(t), \quad x^{k;N+1}(t) = 0, \quad \dot{z}_{N+1}(t) = L_{N+1} = 0 \end{aligned} \quad (9)$$

with

$$L_j(t) := L(t + (j-1)\tau, x^{0;j}(t), \dots, x^{n;j}(t), x^{0;j-1}(t), \dots, x^{n;j-1}(t), z_j(t)).$$

Note that the index k is related to the order of the derivative of x , i is related to the i th subinterval of $[-\tau, N\tau]$, and j is related to the j th subinterval of $[0, (N+1)\tau]$. Consequently, the higher-order problem of Herglotz with time delay (\mathbf{H}_τ^n) can be written as a first-order optimal control problem without time delay as follows:

$$\begin{aligned} z_N(\tau) &\longrightarrow \text{extr}, \quad \text{subject to} \\ \begin{cases} \dot{x}^{k;i}(t) = x^{k+1;i}(t), \\ x^{k;N+1}(t) = 0, \\ \dot{z}_j(t) = L_j(t), \\ \dot{z}_{N+1}(t) = L_{N+1}(t) = 0 \end{cases} \end{aligned} \quad (10)$$

for all $t \in [0, \tau]$ and with the initial conditions

$$\begin{aligned} x^{k;0}(0) &= \mu^{(k)}(-\tau), \quad x^{k;i}(0) = x^{k;i-1}(\tau), \\ z_1(0) &= \gamma, \quad \gamma \in \mathbb{R}, \quad z_j(0) = z_{j-1}(\tau) \end{aligned}$$

for $k = 0, \dots, n-1$, $i = 0, \dots, N$ and $j = 1, \dots, N$. In this form, we look to $x^{k;i}$ and z_j as state variables and to $u_i := x^{n;i}$ as the control variables.

Remark 3. In our previous reduction, we considered the simplest case where $b = N\tau$. If b is not an integer multiple of τ , then there is an integer N such that $(N-1)\tau < b < N\tau$. In that case, the only modification required in the change of variables indicated in (9) is to consider the variables $x^{k;N}$, $k = 0, \dots, n$, and \dot{z}_N as defined in (9) for $t \in [0, b - (N-1)\tau]$ and zero for $t \in]b - (N-1)\tau, \tau]$. Note that with this minor change, the function to be extremized remains the same and, therefore, we can consider that $b = N\tau$.

Remark 4 (Semi-invariance of problem (\mathbf{H}_τ^n) under a group of symmetries). If there is a C^1 transformation group

$$g : [a, b] \times \mathbb{R}^{2m(n+1)+1} \times \mathbb{R}^{d(q+1)} \rightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}, \quad (11)$$

$$g(\alpha(t)) = (\mathsf{T}(\alpha(t)), \mathsf{X}(\alpha(t)), \mathsf{Z}(\alpha(t))),$$

where $\alpha(t)$ stands for

$$(t, x(t), \dot{x}(t), \dots, x^{(n)}(t), x(t-\tau), \dot{x}(t-\tau), \dots, x^{(n)}(t-\tau), z(t), p(t), \dot{p}(t), \dots, p^{(q)}(t)),$$

which for $p(t) = \dot{p}(t) = \dots = p^{(q)}(t) = 0$ coincides with the identity transformation for all $(t, x, z) \in [a-\tau, b] \times \mathbb{R}^m \times \mathbb{R}$, and such that problem (\mathbf{H}_τ^n) satisfies the two equations

$$\frac{z(b)}{b-a} + \frac{d}{dt}F(\alpha(t)) = \frac{Z(\alpha(b))}{T(\alpha(b)) - T(\alpha(a))} \frac{d}{dt}\mathsf{T}(\alpha(t)) \quad (12)$$

and

$$\frac{d}{dt}Z(\alpha(t)) = L(g(\alpha(t))) \frac{d}{dt}\mathsf{T}(\alpha(t)) \quad (13)$$

for some function F of class C^1 , where

$$\frac{d}{d\mathsf{T}}\mathsf{X}(\alpha(t)) = \frac{\frac{d}{dt}\mathsf{X}(\alpha(t))}{\frac{d}{dt}\mathsf{T}(\alpha(t))} \text{ and } \frac{d^k}{d\mathsf{T}^k}\mathsf{X}(\alpha(t)) = \frac{\frac{d}{dt}\left(\frac{d^{k-1}}{d\mathsf{T}^{k-1}}\mathsf{X}(\alpha(t))\right)}{\frac{d}{dt}\mathsf{T}(\alpha(t))},$$

$k = 2, \dots, n$, then problem (\mathbf{H}_τ^n) is semi-invariant under a group of symmetries as in Definition 2.6.

We are now in a position to formulate and prove our main result.

Theorem 3.1 (Noether's second theorem for problem (\mathbf{H}_τ^n)). *If problem (\mathbf{H}_τ^n) is semi-invariant under a group of symmetries (11), that is, if (12)–(13) holds, then there are $d(q+1)$ Noether currents of the form*

$$\begin{aligned} & \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \frac{z(b)}{b-a} \\ & + \sum_{k=1}^n \phi_k(t) \cdot \left. \frac{\partial}{\partial p_J^{(I)}} \left(\frac{d^{k-1}}{d\mathsf{T}^{k-1}} \mathsf{X}(\alpha(t)) \right) \right|_0 + \psi_z(t) \cdot \left. \frac{\partial Z(\alpha(t))}{\partial p_J^{(I)}} \right|_0 \\ & - H(t, x(t), \dot{x}(t), \dots, x^{(n)}(t), z(t), \phi_1(t), \dots, \phi_n(t), \psi_z(t)) \left. \frac{\partial \mathsf{T}(\alpha(t))}{\partial p_J^{(I)}} \right|_0, \end{aligned}$$

$t \in [a, b]$, for $I = 0, \dots, q$, $J = 1, \dots, d$, and $\theta_J^I \in \mathbb{R}^d$, where ψ_z, ϕ_k and H are defined, respectively, in (5)–(7) and $(*)|_0$ stands for $(*)|_{p(t)=\dot{p}(t)=\dots=p^{(q)}(t)=0}$.

Proof. In order to prove the result, we start by considering problem (\mathbf{H}_τ^n) in its optimal control and non-delayed form (10). First, we prove that if (\mathbf{H}_τ^n) is semi-invariant under a group of symmetries, that is, if there exists a C^1 transformation group (11) satisfying (12)–(13), then the non-delayed optimal control problem (10) is invariant in the sense of Definition 2.6. Observe that (12) is equivalent to

$$\frac{z_N(\tau)}{\tau} + \frac{d}{dt}\tilde{F}(\alpha(t)) = \frac{Z_N(\alpha(\tau))}{T(\alpha(\tau))} \frac{d}{dt}T(\alpha(t)), \quad (14)$$

where \tilde{F} is defined for all $t \in [0, \tau]$ by $\tilde{F}(\alpha)(t) := N \cdot F(\alpha)(t)$. Now, defining

$$\begin{aligned} X_{k;i}(\alpha(t)) &:= \frac{d^k}{d\mathbf{T}^k} X(\alpha(t + (i-1)\tau)), \\ T_i(\alpha(t)) &:= T(\alpha(t + (i-1)\tau)), \\ Z_j(\alpha(t)) &:= Z(\alpha(t + (j-1)\tau)) \end{aligned}$$

for fixed $t \in [0, \tau]$, we have

$$\frac{d}{dt}X_{k;i}(\alpha(t)) = X_{k+1;i}(\alpha(t)) \frac{d}{dt}T_i(\alpha(t)) \quad (15)$$

and

$$\frac{d}{dt}Z_j(\alpha(t)) = L_j(g(\alpha(t))) \frac{d}{dt}T_j(\alpha(t)), \quad (16)$$

for $k = 0, \dots, n-1$, $i = 0, \dots, N$, and $j = 1, \dots, N$. From (14)–(16), we conclude that the non-delayed optimal control problem (10) is semi-invariant in the sense of Definition 2.6. This kind of semi-invariance is the required condition for application of the second Noether theorem for optimal control (Theorem 2.7), which asserts the existence of $d(q+1)$ Noether currents of the form

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \frac{z_N(\tau)}{\tau} + \sum_{k=1}^n \sum_{i=0}^N \phi_{k;i}(t) \cdot \left. \frac{\partial X_{k-1;i}(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \sum_{j=1}^N \psi_j(t) \cdot \left. \frac{\partial Z_j(\alpha(t))}{\partial p_J^{(I)}} \right|_0 \\ - \left[\sum_{k=1}^n \sum_{i=0}^N \phi_{k;i}(t) \cdot x^{k;i}(t) + \sum_{j=1}^N \psi_j(t) L_j(t) \right] \left. \frac{\partial T(\alpha(t))}{\partial p_J^{(I)}} \right|_0, \end{aligned}$$

$t \in [0, \tau]$, for $I = 0, \dots, q$, $J = 1, \dots, d$, where $\phi_{k;i}$ and ψ_j are defined from (5)–(6):

$$\phi_{k;i}(t) = \phi_k(t + (i-1)\tau) \text{ and } \psi_j(t) = \psi_z(t + (i-1)\tau),$$

for $i = 0, \dots, N$ and $j = 1, \dots, N$. Finally, we rewrite the result in the original variables, obtaining that there are $d(q+1)$ Noether currents of the form

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \frac{z(b)}{b-a} + \sum_{k=1}^n \phi_k(t) \cdot \left. \frac{\partial X_k(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \psi_z(t) \cdot \left. \frac{\partial Z(\alpha(t))}{\partial p_J^{(I)}} \right|_0 \\ - H(t, x(t), \dot{x}(t), \dots, x^{(n)}(t), z(t), \phi_1(t), \dots, \phi_n(t), \psi_z(t)) \left. \frac{\partial T(\alpha(t))}{\partial p_J^{(I)}} \right|_0. \end{aligned}$$

This concludes the proof. \square

Our result is new even for first-order generalized variational problems.

Corollary 1. *If the first-order problem of Herglotz with time delay*

$$\begin{aligned} z(b) &\longrightarrow \text{extr}, \\ \dot{z}(t) &= L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau), z(t)), \quad t \in [a, b], \\ z(a) &= \gamma \in \mathbb{R}, \quad x(t) = \mu(t), \quad t \in [a-\tau, a], \end{aligned}$$

where μ is a given piecewise initial function, is semi-invariant, then there exist $d(q+1)$ Noether currents of the form

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \frac{z(b)}{b-a} + \phi_1(t) \cdot \left. \frac{\partial X(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \psi_z(t) \cdot \left. \frac{\partial Z(\alpha(t))}{\partial p_J^{(I)}} \right|_0 \\ - [\phi_1(t)\dot{x}(t) + \psi_z(t)L[x; z]_\tau^1(t)] \left. \frac{\partial T(\alpha(t))}{\partial p_J^{(I)}} \right|_0, \end{aligned}$$

$t \in [a, b]$, for $I = 0, \dots, q$, $J = 1, \dots, d$, where ϕ_1 is given by (6) and ψ_z by (5).

Proof. Consider Theorem 3.1 with $n = 1$. \square

As a corollary of Corollary 1, we obtain a new result for delayed classical problems of the Calculus of Variations.

Corollary 2. *If the first-order variational problem with time delay*

$$\int_a^b L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)) dt \longrightarrow \text{extr},$$

with $x(t) = \mu(t)$, $t \in [a-\tau, a]$, for a given piecewise initial function μ , is semi-invariant, then there exists $d(q+1)$ Noether currents of the form

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \phi_1(t) \cdot \left. \frac{\partial X(\alpha(t))}{\partial p_J^{(I)}} \right|_0 + \theta_J^I \frac{z(b)}{b-a} \\ - \left[\phi_1(t)\dot{x}(t) + L(t, x(t), \dot{x}(t), x(t-\tau), \dot{x}(t-\tau)) \right] \left. \frac{\partial T(\alpha(t))}{\partial p_J^{(I)}} \right|_0, \end{aligned}$$

$t \in [a, b]$, for $I = 0, \dots, q$, $J = 1, \dots, d$, where ϕ_1 is given by (6).

Proof. Consider Corollary 1 with L not depending on z . \square

4. Example. In order to illustrate our results, we present a simple example that cannot be covered using available results in the literature. Consider an arbitrary interval $[a, b]$ and let $\tau \in \mathbb{R}$ be a nonnegative real number such that $\tau < b - a$. We address the following problem with $m = d = q = 1$:

$$\begin{aligned} z(b) &\rightarrow \text{extr}, \\ \dot{z}(t) &= x(t-\tau)z(t), \quad t \in [a, b], \\ \text{subject to } z(a) &= \gamma, \quad x(t) = \mu(t), \quad t \in [a-\tau, a], \end{aligned} \tag{17}$$

where $\mu \in PC^1([a-\tau, a]; \mathbb{R})$ is a given initial function. Let p be a $C^1([a, b]; \mathbb{R})$ function and consider the C^1 group of symmetries

$$g(\alpha(t)) = \left(t + p(t), \frac{x(t-\tau)}{1 + \dot{p}(t)}, z(t) \right),$$

that is,

$$\begin{aligned} \mathbb{T}(\alpha(t)) &= T(t, p(t)) = t + p(t), \\ \mathbb{X}(\alpha(t)) &= X(x(t - \tau), \dot{p}(t)) = \frac{x(t - \tau)}{1 + \dot{p}(t)}, \\ \mathbb{Z}(\alpha(t)) &= Z(z(t)) = z(t), \end{aligned}$$

which for $p(t) = \dot{p}(t) = 0$, $t \in [a, b]$, reduce to the identity transformations. Observe that the problem under study is semi-invariant. Indeed, (12) is verified with

$$F(t) = \frac{z(b)}{b - a + p(b) - p(a)} (t + p(t)) - \frac{z(b)}{b - a} t$$

and (13) is also valid because

$$\frac{d}{dt} \mathbb{Z}(\alpha(t)) = \dot{z}(t) = \frac{x(t - \tau)}{1 + \dot{p}(t)} z(t) (1 + \dot{p}(t)) = L(g(\alpha(t))) \frac{d}{dt} \mathbb{T}(\alpha(t)).$$

From Theorem 3.1, we have that there are two Noether currents of the form

$$\begin{aligned} \left. \frac{\partial F(\alpha(t))}{\partial p^{(I)}} \right|_0 + \theta^I \frac{z(b)}{b - a} + \phi_1(t) \cdot \left. \frac{\partial \mathbb{X}(\alpha(t))}{\partial p^{(I)}} \right|_0 + \psi_z(t) \cdot \left. \frac{\partial \mathbb{Z}(\alpha(t))}{\partial p^{(I)}} \right|_0 \\ - [\phi_1(t) \dot{x}(t) + \psi_z(t) L[x; z]_\tau^1(t)] \left. \frac{\partial \mathbb{T}(\alpha(t))}{\partial p^{(I)}} \right|_0, \quad I = 0, 1. \end{aligned}$$

Noting that $\phi_1(t) = 0$ and $\psi_z(t) = e^{\int_t^b x(s - \tau) ds}$, $t \in [a, b]$, the second Noether current reduces to a constant while the first gives a nontrivial conclusion: it asserts that

$$x(t - \tau) z(t) e^{\int_t^b x(s - \tau) ds}$$

is constant along the extremals of problem (17).

5. Concluding remarks. We have deduced new necessary conditions for higher-order generalized variational problems with time delay that are semi-invariant under a group of transformations that depends on arbitrary functions. The conditions are potentially useful, because for many variational problems, the Euler–Lagrange equations and transversality conditions are not enough to obtain an explicit solution. Our main result is new even for classical delayed variational problems.

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